# An Integral Equation Technique for the Exterior and Interior Neumann Problem in Toroidal Regions

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An integral equation technique for the Neumann problem of finding a function $\Phi$ satisfying $\Delta\Phi = 0$ with prescribed values of $\partial\Phi/\partial n$ on the boundary is described. Fourier representation of the potential $\Phi$ on the boundary with respect to two angle-like variables transforms the integral equation to an infinite set of linear equations for the Fourier coefficients of $\Phi$. The singularity of the Green's function is treated by a regularization method: a function with the same singularity is subtracted and its analytically calculated Fourier-transform is added to the Fourier transformed integral equation. A computer code named NESTOR is developed. Applications include studies of toroidal magnetic vacuum fields and calculation of the vacuum field contribution for the 3D free-boundary equilibrium problem.   © 1986 Academic Press, Inc.

## 1. INTRODUCTION

Let $D$ denote the interior and $D^c$ the exterior of a toroidal region with the boundary $\partial D$ and let $\mathbf{B}_0$ be a magnetic field

$$\operatorname{curl} \mathbf{B}_0 = \mathbf{j}, \qquad \operatorname{div} \mathbf{B}_0 = 0 \tag{1.1}$$

produced by a current $\mathbf{j}$ in $D \cup D^c$ (Fig. 1). There then exists a magnetic vacuum field $\mathbf{B} = \nabla\Phi$ in $D$ such that the superposition of the two fields is tangential on the boundary $\partial D$,

$$(\mathbf{B}_0 + \nabla\Phi) \cdot \mathbf{n} = 0, \tag{1.2}$$

and that the potential $\Phi$ is single valued ($\mathbf{n} = $ the exterior normal to $\partial D$). The potential function $\Phi$ is the solution of the interior Neumann problem: it satisfies the Laplace equation $\Delta\Phi = 0$ in $D$ and the normal derivative $\partial\Phi/\partial n$ takes given values on the boundary $\partial D$. The solution is unique up to a constant [1].

By means of Green's theorem the Laplace equation for $\Phi$ can be converted into an integral equation

$$\Phi(\mathbf{x}) + \frac{1}{2\pi}\int_{\partial D} df' \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \Phi(\mathbf{x}') = \frac{1}{2\pi}\int_{\partial D} df' G(\mathbf{x}, \mathbf{x}') \frac{\partial \Phi(\mathbf{x}')}{\partial n'}, \tag{1.3}$$
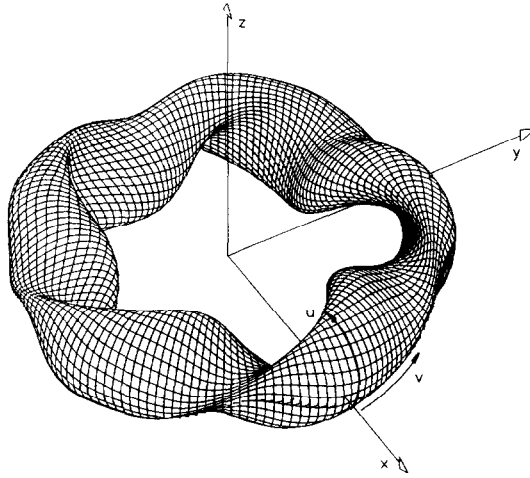
FIG. 1. View of toroidal domain.

where $\mathbf{x}$, $\mathbf{x}'$ are points on the boundary $\partial D$, and $G(\mathbf{x}, \mathbf{x}') = 1/|\mathbf{x} - \mathbf{x}'|$ is the Green's function. With condition (1.2) the right-hand side of the integral equation is known and is considered to be a source term.

The solution of the integral equation gives the potential $\Phi$ on the boundary. The potential $\Phi$ in the interior of the toroidal region is then obtained from the potential and its normal derivative on the boundary by

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi} \int_{\partial D} df' \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \Phi(\mathbf{x}') + \frac{1}{4\pi} \int_{\partial D} df' G(\mathbf{x}, \mathbf{x}') \frac{\partial \Phi(\mathbf{x}')}{\partial n'}, \qquad (1.4)$$

where $\mathbf{x}$ are points in the interior of the region $D$.

A potential function $\Phi$ is obtained in exactly the same way by solving the exterior Neumann problem: it satisfies the Laplace equation in $D^c$ and the condition (1.2) on the boundary $\partial D$. The potential function $\Phi$ is unique up to a constant.

Integral equation techniques for solving the Laplace equation have been applied in different fields. For example, the incompressible potential flow around three-dimensional bodies has been treated [2]. For other applications see [3, 4, 5]. Most applications concern simply connected boundaries.

In toroidal geometry the solution of the integral equation is given by Martensen for axisymmetric boundaries [6]. In the 2D ERATO code for stability analysis of axisymmetric equilibria, the same Green's function method is applied to calculate the vacuum energy contribution [7]. The HERA code, an extended version of ERATO for stability analysis of helically symmetric equilibria, also provides for calculation of the vacuum contribution by solving the integral equation [8, 9].

The present paper presents an integral equation method for general toroidal geometry. Two applications appear to be of interest.

As solution of the exterior Neumann problem, the vacuum field contribution in the 3D MHD equilibrium problem can be calculated to study the free-boundary equilibrium, including the field produced by external currents. In the BETA code [10] the vacuum energy problem is treated as a variational problem by considering the Dirichlet integral $\int (\nabla \Phi)^2 \, dV$. The discrete approximation leads to difference equations in the vacuum region and therefore requires a second outer control surface to keep the vacuum region finite. By solving the integral equation on the boundary $\partial D$ one gets a solution for the Neumann problem in the infinite region exterior to the plasma surface $\partial D$.

As solution of the interior Neumann problem, toroidal vacuum fields can be generated with a magnetic surface on the boundary. There are conjectures that for appropriate boundaries it is possible to find configurations with "good" magnetic surfaces in the whole region. The problem of finding such configurations has been tackled by various methods: superposing special harmonic functions [11]; removing islands by appropriate correction fields [12]; and also solving the boundary-value problem by difference methods for Laplace's equation [13], but only for a restricted class of Heliac-type boundaries.

Here, the integral equation is treated for general three-dimensional toroidal surfaces. Because of the toroidal geometry it appears appropriate to introduce angle-like variables $u$ and $v$. The surface $\partial D$ and the potential function $\Phi$ are represented by Fourier series of $u$ and $v$. Then, from the Fourier transform of the integral equation, one gets an infinite set of linear equations for the Fourier coefficients of the potential $\Phi$. The equations are solved approximately by neglecting higher harmonics.

The singularity of the Green's function and its normal derivative are treated by the following regularization method: functions are introduced with analytically calculable Fourier transform and the same singular behaviour as the Green's function and its normal derivative. They are subtracted, the nonsingular difference is Fourier-transformed by a standard numerical method and the analytically calculated singular part is added again.

## 2. SOLUTION OF THE INTEGRAL EQUATION

The interior Neumann problem for a toroidal region $D$ with the boundary $\partial D$ is considered. The boundary $\partial D$ is assumed to consist of $n_p$ toroidal periods, and the normal derivative $\partial \Phi / \partial n$ on the boundary is assumed to have the same periodicity. With angle-like variables $u$ and $v$ introduced, one period of the boundary is given by mapping the unit square $0 \leqslant u < 1$, $0 \leqslant v < 1$ onto the surface $\partial D$,

$$r = \sum_{m=-m_b, n=-n_b}^{m_b, n_b} \hat{r}_{mn} e^{2\pi i(mu+nv)}, \qquad \hat{r}_{mn}^* = \hat{r}_{-m-n},$$

$$z = \sum_{m=-m_b, n=-n_b}^{m_b, n_b} \hat{z}_{mn} e^{2\pi i(mu+nv)}, \qquad \hat{z}_{mn}^* = \hat{z}_{-m-n}, \qquad (2.1)$$

$$\phi = \frac{2\pi}{n_p} v,$$

where $(r, \phi, z)$ are cylindrical coordinates.

With $u$ and $v$ as independent variables, the integral equation (1.3) takes the form

$$\Phi(u, v) + \int_0^1 \int_0^1 du' \, dv' g(u, v, u', v') \, \Phi(u', v') = \int_0^1 \int_0^1 du' \, dv' h(u, v, u', v'), \quad (2.2)$$

where the kernel is given by

$$g(u, v, u', v') = \frac{1}{2\pi} \sum_{l=1}^{n_p} \frac{(\mathbf{x} - \mathbf{x}'^{(l)}) \cdot [\mathbf{x}_{u'}'^{(l)} \times \mathbf{x}_{v'}'^{(l)}]}{|\mathbf{x} - \mathbf{x}'^{(l)}|^3} \qquad (2.3)$$

and the source term is defined by

$$\cdot \, h(u, v, u', v') = -\frac{1}{2\pi} \sum_{l=1}^{n_p} \frac{\mathbf{B}_0' \cdot [\mathbf{x}_{u'}'^{(l)} \times \mathbf{x}_{v'}'^{(l)}]}{|\mathbf{x} - \mathbf{x}'^{(l)}|}. \qquad (2.4)$$

In terms of Cartesian coordinates the indexed vectors $\mathbf{x}^{(l)}$ are defined by

$$\mathbf{x}^{(l)} = \begin{pmatrix} r(u, v) \cdot \cos(2\pi(l-1+v)/n_p) \\ r(u, v) \cdot \sin(2\pi(l-1+v)/n_p) \\ z(u, v) \end{pmatrix}, \qquad l = 1, ..., n_p, \qquad (2.5)$$

and $\mathbf{x}_u^{(l)}$, $\mathbf{x}_v^{(l)}$ are the derivatives with respect to $u$ and $v$ resp.: $\mathbf{x}_u := (\partial/\partial u) \, \mathbf{x}$. The vector $\mathbf{x}$ is identical with $\mathbf{x} \equiv \mathbf{x}^{(1)}$.

The potential $\Phi(u, v)$ is a periodic function of $u$ and $v$. It is therefore reasonable to express $\Phi$ as a Fourier series:

$$\Phi(u, v) = \sum_{m=-\infty, n=-\infty}^{\infty, \infty} \hat{\Phi}_{mn} e^{2\pi i(mu+nv)} \qquad (2.6)$$

with the reality condition $\hat{\Phi}_{mn}^* = \hat{\Phi}_{-m-n}$.

With this ansatz substituted in the integral equation (2.2), the Fourier transform with respect to $u$ and $v$ leads to an infinite set of linear equations for the Fourier coefficients $\hat{\Phi}_{mn}$,

$$\hat{\Phi}_{mn} + \sum_{m'=-\infty, n'=-\infty}^{\infty, \infty} \hat{g}_{mnm'n'} \hat{\Phi}_{m'n'} = \hat{h}_{mn}, \qquad (2.7)$$

where the matrix elements $\hat{g}_{mnm'n'}$ and the source term elements $\hat{h}_{mn}$ are

$$\hat{g}_{mnm'n'} = \int_0^1 \int_0^1 \int_0^1 \int_0^1 du'\, dv'\, du\, dv\, g(u, v, u', v')\, e^{2\pi i(m'u' + n'v' - mu - nv)} \qquad (2.8)$$

and

$$\hat{h}_{mn} = \int_0^1 \int_0^1 \int_0^1 \int_0^1 du'\, dv'\, du\, dv\, h(u, v, u', v')\, e^{-2\pi i(mu + nv)}. \qquad (2.9)$$

The main difficulty consists in the calculation of the Fourier transforms (Eqs. (2.8) and (2.9)). Because of the singularity in $g(u, v, u', v')$ and $h(u, v, u', v')$ at $|x - x'| = 0$ it is not possible to use the standard numerical Fourier transform methods.

A regularization procedure therefore has to be applied: one has to find a function with the same singular behaviour which can be analytically Fourier-transformed with respect to $u$ and $v$. With $|x - x'|$ expanded for fixed values of $u'$ and $v'$ at the singular point, the singular part of the Green's function takes the form

$$h \approx \frac{\mathbf{B}_0' \cdot [\mathbf{x}_{u'}'^{(1)} \times \mathbf{x}_{v'}'^{(1)}]}{(\mathbf{x}_{u'}'^2 (u - u')^2 + 2\mathbf{x}_{u'}' \mathbf{x}_{v'}'(u - u')(v - v') + \mathbf{x}_{v'}'^2 (v - v')^2)^{1/2}},$$

where $\mathbf{x}_{u'}'$, $\mathbf{x}_{v'}'$ are regular for all values of $u'$, $v'$. A similar behaviour follows for $g$.

The following periodic functions are introduced:

$h^{\text{sing}}(u - u', v - v', u', v')$

$$= \frac{F}{2(a \tan^2 \pi(u - u') + 2b \tan \pi(u - u') \tan \pi(v - v') + c \tan^2 \pi(v - v'))^{1/2}} \qquad (2.10)$$

and

$g^{\text{sing}}(u - u', v - v', u', v')$

$$= \frac{A \tan^2 \pi(u - u') + 2B \tan \pi(u - u') \tan \pi(v - v') + C \tan^2 \pi(v - v')}{2(a \tan^2 \pi(u - u') + 2b \tan \pi(u - u') \tan \pi(v - v') + c \tan^2 \pi(v - v'))^{3/2}} \qquad (2.11)$$

with the coefficients

$$a = \mathbf{x}_{u'}'^2, \qquad b = \mathbf{x}_{u'}' \mathbf{x}_{v'}', \qquad c = \mathbf{x}_{v'}'^2 \qquad (2.12)$$

and

$$\begin{aligned}
A &= \tfrac{1}{2}\mathbf{x}_{u'u'}' \cdot [x_{u'}' \times x_{v'}'], \\
B &= \tfrac{1}{2}\mathbf{x}_{u'v'}' \cdot [x_{u'}' \times x_{v'}'], \\
C &= \tfrac{1}{2}\mathbf{x}_{v'v'}' \cdot [x_{u'}' \times x_{v'}'], \\
F &= -\mathbf{B}_0' \cdot [x_{u'}' \times x_{v'}'],
\end{aligned} \qquad (2.13)$$

where $\mathbf{x}_{u'u'}'$, $\mathbf{x}_{u'v'}'$, $\mathbf{x}_{v'v'}'$ are the second derivatives ($\mathbf{x}_{u'u'}' := (\partial^2/\partial u'^2)\, x'$).

If these functions are considered for fixed values of $u'$ and $v'$ as functions of $u$ and $v$, they have the desired singular behaviour and can be analytically Fourier-transformed with respect to the variables $u$ and $v$, shown in the Appendix.

With these analytically calculated integrals, the matrix elements $\hat{g}_{mnm'n'}$ can be written as

$$
\hat{g}_{mnm'n'}
$$
$$
= \int_0^1 \int_0^1 \int_0^1 \int_0^1 du\, dv\, du'\, dv'(g(u, v, u', v')
$$
$$
- g^{\mathrm{sing}}(u - u', v - v', u', v'))\, e^{2\pi i(m'u' + n'v' - mu - nv)}
$$
$$
+ \int_0^1 \int_0^1 du'\, dv'\, \hat{g}_{mn}^{\mathrm{an}}(u', v')\, e^{2\pi i((m' - m)\, u' + (n' - n)\, v')} \tag{2.14}
$$

with

$$
\hat{g}_{mn}^{\mathrm{an}}(u', v') = \int_0^1 \int_0^1 du\, dv\, g^{\mathrm{sing}}(u - u', v - v', u', v')\, e^{-2\pi i(m(u - u') + n(v - v'))}. \tag{2.15}
$$

The integrands in Eq. (2.14) are finite and can be Fourier-transformed by standard numerical methods, while the integral in Eq. (2.15) is calculated analytically (see Appendix). The source term is treated in exactly the same way:

$$
\hat{h}_{mn} = \int_0^1 \int_0^1 \int_0^1 \int_0^1 du\, dv\, du'\, dv'(h(u, v, u', v') - h^{\mathrm{sing}}(u - u', v - v', u', v'))\, e^{-2\pi i(mu + mv)}
$$
$$
+ \int_0^1 \int_0^1 du'\, dv'\, \hat{h}_{mn}^{\mathrm{an}}(u', v')\, e^{-2\pi i(mu' + nv')} \tag{2.16}
$$

with

$$
\hat{h}_{mn}^{\mathrm{an}}(u', v') = \int_0^1 \int_0^1 du'\, dv'\, h^{\mathrm{sing}}(u - u', v - v', u', v')\, e^{-2\pi i(m(u - u') + n(v - v'))}. \tag{2.17}
$$

These equations are derived for the interior Neumann problem. One gets the solution for the exterior Neumann problem by changing the sign of the surface integrals in the integral equations (2.2).

## 3. Numerical Approximation and Results

One purpose of applying the solution method to the interior Neumann problem is to find toroidal vacuum fields with "good" magnetic surfaces and other desired properties, e.g., a magnetic well and a prescribed slope of the rotational transform.

These solutions can serve as suitable starting configurations to study stellarator plasma equilibria.

The field produced by a line current along the $z$-axis is chosen as external magnetic field $\mathbf{B}_0$,

$$\mathbf{B}_0 \equiv (B_r, B_\phi, B_z) = \left(0, \frac{I}{r}, 0\right). \tag{3.1}$$

The boundary coordinates are given as Fourier series (Eq. (2.1)) of $u$ and $v$. For stellarator configuration studies one usually considers fields with a symmetry which reads for the vacuum potential $\Phi(r, -\phi, -z) = -\Phi(r, \phi, z)$ and for the boundary coordinates $r(-u, -v) = r(u, v)$, $z(-u, -v) = -z(u, v)$. In this particular case the generally complex matrix elements $\hat{g}_{mnm'n'}$ become real, and the $\hat{h}_{mn}$ and the Fourier coefficients of the potential $\hat{\Phi}_{mn}$ are purely imaginary.

The integral equation is approximated by truncating the infinite set of equations (2.7). Choosing appropriate integers $M$ and $N$ and neglecting all Fourier harmonics $m$, $n$ with $|m| > M$ and $|n| > N$, one obtains the Fourier components of the potential $\hat{\Phi}_{mn}$ by

$$\hat{\Phi}_{mn} + \sum_{m'=-M,\,n'=-N}^{M,\,N} \hat{g}_{mnm'n'} \hat{\Phi}_{m'n'} = \hat{h}_{mn} \qquad \text{for} \quad |m| \leqslant M, |n| \leqslant N. \tag{3.2}$$

The matrix elements $\hat{g}_{mnm'n'}$ (Eq. (2.8)) and $\hat{h}_{mn}$ (Eq. (2.9)), which are four-dimensional Fourier integrals, have to be calculated numerically. If an equidistant mesh in the $u, v$ unit square with $N_u$ and $N_v$ intervals is introduced, the mesh points are given by

$$u_j = \frac{j}{N_u}, \qquad j = 0, N_u - 1,$$

$$v_k = \frac{k}{N_v}, \qquad k = 0, N_v - 1. \tag{3.3}$$

A Fourier integral is approximated by the discrete Fourier transform [14, 15] defined here for the potential

$$\hat{\Phi}_{mn} = \frac{1}{N_u N_v} \sum_{j=0,\,k=0}^{N_u-1,\,N_v-1} \Phi\left(\frac{j}{N_u}, \frac{k}{N_v}\right) e^{2\pi i \left(\frac{jm}{N_u} + \frac{kn}{N_v}\right)}. \tag{3.4}$$

To get an approximation for Fourier components $|m| \leqslant M$ and $|n| \leqslant N$, the number of mesh points $N_u$, $N_v$ is chosen so that $N_u \geqslant 2(M+1)$, $N_v \geqslant 2(N+1)$.

The error of the numerical solution for the Fourier harmonics $\hat{\Phi}_{mn}$ is composed of the discretization error due to the discretization in real space $(N_u, N_v)$ and the error due to the truncation $(M, N)$ of the infinite set of equations for the $\hat{\Phi}_{mn}$. It can be advantageous to separate these two contributions by using a fine $N_u$, $N_v$ mesh even if a small number of harmonics $\hat{\Phi}_{mn}$ are taken into account.

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The amount of numerical work is essentially determined by the number of $N_u$, $N_v$ mesh points in real space and is proportional to $N_u^2 N_v^2$. The computation time for a case with $N_u = 32$, $N_v = 32$, $M = 10$, $N = 6$ is about 5 sec on a CRAY-1.

The magnetic field in the region $D$ is then obtained from Eq. (1.4),

$$\mathbf{B} = \mathbf{B}_0 + \frac{1}{4\pi} \int \nabla G(\mathbf{x}, \mathbf{x}')(\mathbf{B}_0' \cdot d\mathbf{f}') - \frac{1}{4\pi} \int \Phi(\mathbf{x}') \nabla (\nabla' G(\mathbf{x}, \mathbf{x}') \cdot d\mathbf{f}'), \qquad (3.5)$$

or, introducing the surface current density

$$\mathbf{j} = -\frac{1}{4\pi} [\mathbf{n} \times (\mathbf{B}_0 + \nabla \Phi)], \qquad (3.6)$$

one can transform Eq. (3.5) into the Biot–Savart formula

$$\mathbf{B} = -\int_{\partial D} d\mathbf{f}' \mathbf{j}' \times \nabla G(\mathbf{x}, \mathbf{x}'). \qquad (3.7)$$

For numerical field calculations the former representation of $\mathbf{B}$ (Eq. (3.5)) has the advantage that also the discretized form of Eq. (3.5) exactly satisfies div $\mathbf{B} = 0$ and curl $\mathbf{B} = 0$, because of $\Delta G(\mathbf{x}, \mathbf{x}') = 0$ and curl $\nabla G(\mathbf{x}, \mathbf{x}') = 0$. The integrals are discretized with respect to the integration variable $\mathbf{x}$. The differentiation with respect to the unprimed variable is therefore not affected.

The NESTOR (*NE*umann *S*olver for *TO*roidal *R*egions) computer code was developed [16] for solving the integral equation by the above-described technique. Some applications to the interior Neumann problem are presented.

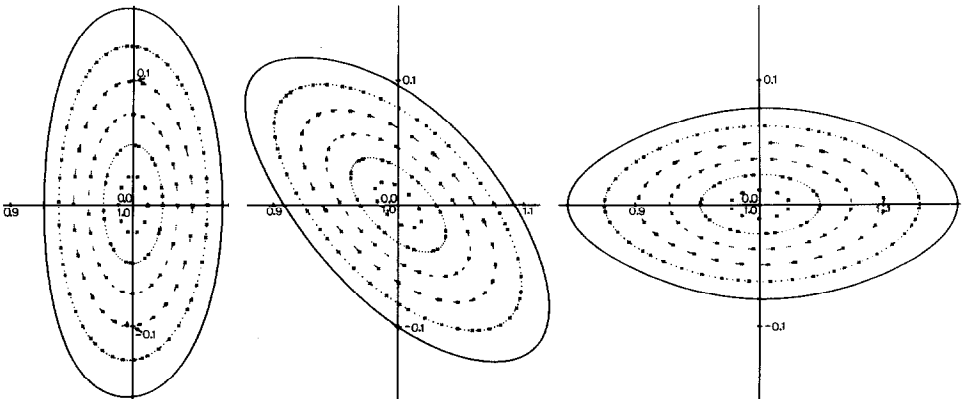Figure 2 shows a comparison of an analytically calculated $l = 2$ stellarator



FIG. 2. The Poincaré plot of an analytically calculated $l = 2$ stellarator field ($\cdots$) is compared with the numerical solution of the boundary-value problem ($\times \times \times$). Discretization of boundary (—) $N_u = 32$, $N_v = 32$ poloidal and toroidal mesh points, number of toroidal periods $n_p = 5$, number of Fourier harmonics $\Phi_{mn}$ taken into account $M = 10$, $N = 5$.

vacuum field and the numerical solution of the corresponding boundary-value problem. This field is produced by a special choice of Dommaschk potentials [17],

$$\Phi(r, \phi, z) = \phi + \left( a \frac{\partial}{\partial z} + \frac{b}{m} \frac{\partial}{\partial \phi} \right)$$
$$\times \left( \left( \left( \frac{z^3 r}{6} \frac{\partial}{\partial r} C^N_{m,0} + zr^3 \frac{\partial}{\partial r} \left( \frac{1}{r^2} C^N_{m,1} \right) \right) \sin m\phi + \left( \frac{z^2}{2} C^N_{m,0} + C^N_{m,1} \right) \cos m\phi \right)$$

$$(3.8)$$

with

$$C^N_{m,0} = (r^m - r^{-m})/2m,$$
$$C^N_{m,1} = (-(m-1) r^{m+2} + (m+1) r^m$$
$$- (m+1) r^{-m+2} + (m-1) r^{-m})/8m(m^2-1). \tag{3.9}$$

The number of periods is $m = 5$ and the coefficients are $a = -1.495873$, $b = -3.270651$. The configuration has magnetic surfaces. For the numerical calculation one of the outer surfaces is chosen as boundary and a Fourier series representation of that surface is approximately determined. The coordinates of boundaries with stellarator symmetry can be written as

$$r = \sum_{m=0, \, n=-n_b}^{m_b, \, n_b} C^r_{m,n} \cos(mu + nv),$$
$$z = \sum_{m=0, \, n=-n_b}^{m_b, \, n_b} C^z_{m,n} \sin(mu + nv). \tag{3.10}$$

TABLE I

Fourier Coefficients $C^r_{mn}$, $C^z_{mn}$ of the Boundary for the $l = 2$ Stellarator Configuration

| | $M$ $N$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $C^r_{mn}$ | | | | | |
| | -2 | 0.000056 | -0.000067 | 0.000373 | 0.000462 |
| | -1 | -0.000921 | -0.034645 | 0.000575 | -0.001509 |
| | 0 | 0.997922 | 0.093260 | 0.002916 | 0.001748 |
| | 1 | -0.000921 | 0.000880 | -0.000231 | -0.000239 |
| | 2 | 0.000056 | 0.000178 | 0.000082 | 0.000052 |
| $C^z_{mn}$ | | | | | |
| | -2 | 0.000076 | 0.000069 | -0.000374 | -0.000518 |
| | -1 | 0.000923 | 0.035178 | 0.000257 | 0.002233 |
| | 0 | 0.000000 | 0.099830 | 0.003096 | 0.001828 |
| | 1 | -0.000923 | 0.000860 | 0.000321 | 0.000257 |
| | 2 | -0.000076 | -0.000179 | 0.000007 | 0.000035 |
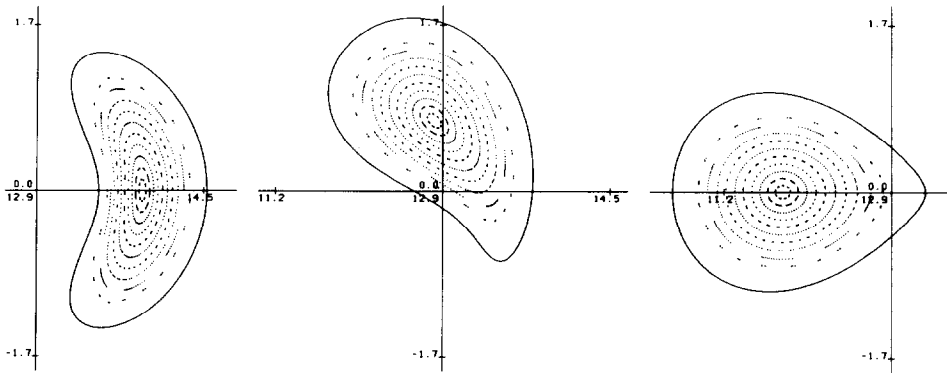
FIG. 3.   Poincaré plot of an $l = 1, 2, 3$ stellarator field [18]. Aspect ratio $A = 13$, number of periods $n_p = 5$, poloidal and toroidal mesh for boundary $N_u = 32$, $N_v = 32$. Number of poloidal and toroidal harmonics taken into account $M = 10$, $N = 6$.

The Fourier coefficients for the $l = 2$ stellarator configuration are given in Table I. With this surface as boundary, the integral equation is solved. Poincaré plots of the magnetic field are shown in Fig. 2. The dots are obtained from the analytical potential (Eq. (3.8)) and the crosses from the numerical solution of the boundary-value problem with the boundary plotted as a solid line. The results are in good agreement.

To illustrate possible applications in Figs. 3 and 4, Poincaré plots are shown for stellarator vacuum fields with $l = 1, 2, 3$ field content. The two configurations differ from each other just in their aspect ratios: $A = 13$ and $A = 4$. The shapes of their boundaries as a function of $\phi$ are kept fixed. The Fourier coefficients of their boundary coordinates (Eq. (3.1)) are given in Table II.
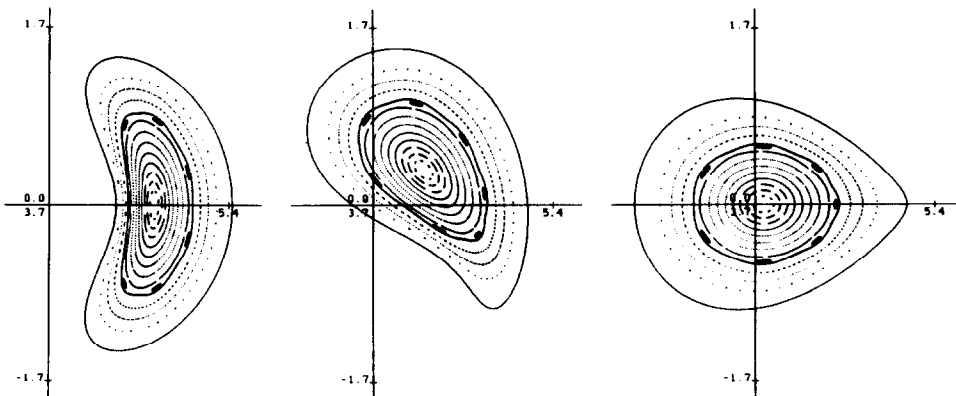


FIG. 4.   Poincaré plot of an $l = 1, 2, 3$ stellarator field. Aspect ratio $A = 4$, number of periods $n_p = 5$, poloidal and toroidal mesh for boundary $N_u = 32$, $N_v = 32$. Number of poloidal and toroidal harmonics taken into account $M = 10$, $N = 6$.

TABLE II

Fourier Coefficients $C^r_{mn}$, $C^z_{mn}$ of the Boundaries for the $l = 1, 2, 3$ Stellarator Configurations with Aspect Ratio $A = 4$ and $A = 13$

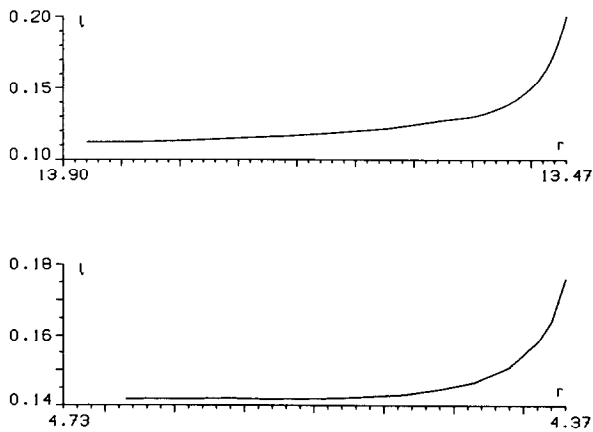| | | M | | |
|---|---|---|---|---|
| | N | 0 | 1 | 2 |
| $A = 13$ | | | | |
| $C^r_{mn}$ | | | | |
| | −2 | 0.0 | 0.0 | 0.07 |
| | −1 | 0.4 | −0.325 | 0.24 |
| | 0 | 12.85 | 0.9 | 0.05 |
| | 1 | 0.4 | −0.035 | 0.0 |
| $C^z_{mn}$ | | | | |
| | −2 | 0.0 | 0.0 | −0.07 |
| | −1 | −0.2 | 0.255 | 0.24 |
| | 0 | 0.0 | 1.1 | 0.05 |
| | 1 | 0.2 | −0.035 | 0.0 |
| $A = 4$ | | | | |
| $C^r_{mn}$ | | | | |
| | −2 | 0.0 | 0.0 | 0.07 |
| | −1 | 0.133 | −0.325 | 0.24 |
| | 0 | 4.283 | 0.9 | 0.05 |
| | 1 | 0.133 | −0.035 | 0.0 |
| $C^z_{mn}$ | | | | |
| | −2 | 0.0 | 0.0 | −0.07 |
| | −1 | −0.067 | 0.255 | 0.24 |
| | 0 | 0.0 | 1.1 | 0.05 |
| | 1 | 0.067 | −0.035 | 0.0 |



FIG. 5. The rotational transform $\iota$ for configurations shown in Figs. 3 and 4 are plotted versus radius from the magnetic axis to the boundary.

The $A = 13$ case shows a set of magnetic surfaces. Islands are not resolved. For the $A = 4$ configuration a chain of islands is resolved, where the rotational transform $\iota$ crosses the rational value $\iota = \frac{1}{7}$. The rotational transform $\iota$ versus radius is shown in Fig. 5 for both configurations.

The finite $\beta$ plasma equilibrium and stability properties of the $A = 13$ case have been studied by Nührenberg and Zille [18]. In the 3D equilibrium codes [10, 19, 20], which assume nested toroidal flux surfaces, island formation is not resolved. It is thus useful to include vacuum field calculations, which allow the analysis of islands at least in the vacuum field. In the present case no dangerous island regions appear.

## 4. CONCLUSIONS

A Green's function method for solving the interior and exterior Neumann problem for toroidal regions is described. The NESTOR computer code for implementing this method was developed and succesfully applied. First results show that the method can be another tool for studying toroidal vacuum field configurations. To apply the method to the free-boundary 3D equilibrium problem, the coupling of NESTOR to the MOMCON 3D equilibrium code [20] is being implemented.

## APPENDIX

For the regularization of the integral equation (1.3) one needs the two-dimensional Fourier transform of singular functions given by

$$I_{mn} = \pi \int_0^1 \int_0^1 du\, dv \frac{e^{2\pi i(mu + nv)}}{(a \tan^2(\pi u) + 2b \tan(\pi u) \tan(\pi v) + c \tan^2(\pi v))^{1/2}}, \tag{A1}$$

$$K_{mn} = \pi \int_0^1 \int_0^1 du\, dv \frac{(A \tan^2(\pi u) + 2B \tan(\pi u) \tan(\pi v) + C \tan^2(\pi v))\, e^{2\pi i(mu + nv)}}{(a \tan^2(\pi u) + 2b \tan(\pi u) \tan(\pi v) + c \tan^2(\pi v))^{3/2}} \tag{A2}$$

with $ac - b^2 > 0$. The integrals $K_{mn}$ can be derived from the integrals $I_{mn}$ by differentiation:

$$K_{mn} = -2 \left( A \frac{\partial}{\partial a} + B \frac{\partial}{\partial b} + C \frac{\partial}{\partial c} \right) I_{mn}. \tag{A3}$$

To compute the $I_{mn}$ a generating function $I$ is introduced:

$$I = \sum_{m=0, n=0}^{\infty, \infty} I_{mn} s^m t^n. \tag{A4}$$

Summing up the power series, one obtains $I$ in closed form:

$$I = \pi \int_0^1 \int_0^1 \frac{du\,dv}{(1 - se^{2\pi iu})(1 - te^{2\pi iv})(a \tan^2(\pi u) + 2b \tan(\pi u) \tan(\pi v) + c \tan^2(\pi v))^{1/2}}.$$
(A5)

Introducing new variables $r, y$

$$y = \tan \pi u,$$

$$ry = \tan \pi v,$$

one get for $I$

$$I = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dr\,dy \left( \frac{1}{y + i\alpha} - \frac{1}{y - i} \right) \left( \frac{1}{ry + i\beta} - \frac{1}{ry - i} \right) \frac{1}{(a + 2br + cr^2)^{1/2}} \quad \text{(A6)}$$

with

$$\alpha = \frac{1 - s}{1 + s},$$

$$\beta = \frac{1 - t}{1 + t}.$$

Integrating $I$ with respect to $y$, one obtains

$$I = \frac{1}{2} \int_0^\infty dr \left( \frac{1}{\beta + \alpha r} + \frac{1}{1 + r} \right) \frac{1}{(a - 2br + cr^2)^{1/2}}$$

$$+ \frac{1}{2} \int_0^\infty dr \left( \frac{1}{1 + \alpha r} + \frac{1}{\beta + r} \right) \frac{1}{(a + 2br + cr^2)^{1/2}}. \quad \text{(A7)}$$

If the integration variable $x = (1 - r)/(1 + r)$ is introduced and $\alpha$ and $\beta$ are again expressed by $s$ and $t$, the function $I$ can be written as a sum of four terms:

$$I = h^+(s, t) + h^+(0, 0) + h^-(s, 0) + h^-(0, t), \quad \text{(A8)}$$

where the function $h$ is defined by

$$h^\pm(s, t) = \frac{(1 + s)(1 + t)}{2} \int_{-1}^{+1} \frac{dx}{(1 - st - (s - t)x)(a^\mp + 2dx + a^\pm x^2)^{1/2}} \quad \text{(A9)}$$

and the coefficients are given by

$$a^+ = a + 2b + c,$$
$$a^- = a - 2b + c, \quad \text{(A10)}$$
$$d = c - a.$$

The final step in calculating the $I_{mn}$ is to expand $I$ again as a power series in $s$ and $t$. One starts by expanding the function $h$

$$h^{\pm}(s, t) = \frac{1}{2} \frac{(1+s)(1+t)}{1-st} \sum_{l=0}^{\infty} \left(\frac{s-t}{1-st}\right)^{l} T_{l}^{\pm} \tag{A11}$$

with

$$T_{l}^{\pm} = \int_{-1}^{+1} dx \frac{x^{l}}{(a^{\mp} + 2dx + a^{\pm}x^{2})^{1/2}}. \tag{A12}$$

Expanding $h$ further, one finally obtains for the coefficients $I_{mn}$

$$I_{mn} = \begin{cases} c_{mn}^{+} + c_{m-1n}^{+} + c_{mn-1}^{+} + c_{m-1n-1}^{+} & \text{for} \quad m \geqslant 1, n \geqslant 1, \\ c_{m0}^{+} + c_{m-10}^{+} + c_{m0}^{-} \quad + c_{m-10}^{-} & \text{for} \quad m \geqslant 1, n = 0, \\ c_{0n}^{+} + c_{0n-1}^{+} + c_{0n}^{-} \quad + c_{0n-1}^{-} & \text{for} \quad m = 0, n \geqslant 1, \\ c_{00}^{+} + c_{00}^{+} \quad + c_{00}^{-} \quad + c_{00}^{-} & \text{for} \quad m = 0, n = 0, \end{cases} \tag{A13}$$

with

$$c_{mn}^{\pm} = \sum_{l=0}^{(m+n-|m-n|)/2} \frac{(-1)^{l+|m-n|-m+n} \left(\frac{m+n+|m-n|}{2} + l\right)! \, T_{|m-n|+2l}^{\pm}}{\left(\frac{m+n-|m-n|}{2} - l\right)! \, (|m-n|+l)! \, l!}. \tag{A14}$$

The functions $T_{l}^{\pm}$ defined in Eq. (A12) can be calculated by using a recurrence relation. Expressing the coefficients $a^{+}$, $a^{-}$, $d$ by $a$, $b$, $c$ (see Eq. (A10)), one obtains

$$
\begin{aligned}
T_{0}^{\pm} &= (1/\sqrt{a \pm 2b + c}) \log(\sqrt{c(a \pm 2b + c)} + c \pm b)/(\sqrt{a(a \pm 2b + c)} - a \mp b), \\
T_{1}^{\pm} &= (1/\sqrt{a \pm 2b + c})(2(\sqrt{c} - \sqrt{a}) - (c-a) T_{0}^{\pm}), \\
T_{l}^{\pm} &= (1/l\sqrt{a \pm 2b + c})(2(\sqrt{c} + (-1)^{l} \sqrt{a}) \\
&\quad - (2l-1)(c-a) T_{l-1}^{\pm} - (l-1)(a \mp 2b + c) T_{l-2}^{\pm}) \qquad \text{for} \quad l \geqslant 2.
\end{aligned}
\tag{A15}
$$

The Fourier integrals $K_{mn}$ follow from the $I_{mn}$ by differentiation (see Eq. (A3)). For the $K_{mn}$ one gets the same formulas, (A13), (A14), as for the $I_{mn}$; only the integrals $T_{l}^{\pm}$ have to be replaced by integrals $S_{l}^{\pm}$, which are given by

$$S_{l}^{\pm} = \int_{-1}^{+1} dx \, x^{l} \frac{(A^{-} + 2Dx + A^{+}x^{2})}{(a^{-} + 2dx + a^{+}x^{2})^{3/2}} \tag{A16}$$

with

$$A^{+} = A + 2B + C,$$
$$A^{-} = A - 2B + C,$$
$$D = C - A.$$

As for the $T_l^{\pm}$, recurrence formulas can be derived for the $S_l^{\pm}$, but the $S_l^{\pm}$ can also be expressed in terms of the $T_l^{\pm}$ by appropriate partial integration. One obtains for $S_l^+$

$$(a + 2b + c)(ac - b^2) \, S_l^+$$
$$= ((A + 2B + C)(ac - b^2) + l(k_1(a + 2b + c) + k_2(c - a))) \, T_l^+$$
$$+ l(k_1(c - a) + k_2(a - 2b + c)) \, T_{l-1}^+$$
$$- \frac{(c + b) \, k_1 + (c - b) \, k_2}{\sqrt{c}} - (-1)^l \frac{(a + b) \, k_1 - (a - b) \, k_2}{\sqrt{a}} \qquad \text{(A17)}$$

with

$$k_1 = 4(a + c) \, B - 4(A + C) \, b,$$
$$k_2 = 4C(a + b) - 4B(c - a) - 4A(c + b).$$

The formula for $S_l^-$ is obtained by replacing $b$ by $-b$ and $B$ by $-B$, and $T_l^+$, $T_{l-1}^+$ with $T_l^-$, $T_{l-1}^-$.

For large $m$ and $n$ an asymptotic expansion can be derived for the Fourier integrals $I_{mn}$ and $K_{mn}$. For $ac - b^2 > 0$ the leading terms are [21]

$$I_{mn} = \frac{1}{(an^2 - 2bnm + cm^2)^{1/2}},$$
$$K_{mn} = \frac{An^2 - 2Bnm + Cm^2}{(an^2 - 2bnm + cm^2)^{3/2}}. \qquad \text{(A18)}$$

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